CONJUGATE PERMUTATIONS IN A_n

ZHIPING XU

Abstract. We know that two permutations in S_n are conjugate if and only if their decompositions consist of the same cycle type. And a conjugacy class in S_n of even permutations is either equal to a single conjugacy class in A_n, or splits into two conjugacy classes in A_n. So two even permutations of the same cycle type may not be conjugate in A_n. In this article we introduce a simple and practicable criterion for determining whether two even permutations are conjugate in A_n.

For convience, we assume that all permutations here have already been decomposed into disjoint cycles.

Let a and b be two conjugate even permutations in S_n, then we can easily compute a permutation $\tau \in S_n$ such that $\tau a \tau^{-1} = b$. Let σ be another permutation (σ may equal to τ) such that $\sigma a \sigma^{-1} = b$, then $\tau^{-1}\sigma a \sigma^{-1}\tau = a$, which means $\tau^{-1}\sigma \in Stab_{S_n}(a)$. Then $\sigma \in \tau Stab_{S_n}(a)$, which means that any σ satisfy $\sigma a \sigma^{-1} = b$ if and only if $\sigma \in \tau Stab_{S_n}(a)$.

We observe that if $Stab_{S_n}(a)$ contains of both odd and even permutations, then there exists an element π of $Stab_{S_n}(a)$ such that $\sigma = \tau\pi$ is even, which implies a and b are conjugate in A_n. And since (1) is in $Stab_{S_n}(a)$, if all elements of $Stab_{S_n}(a)$ have the same parity, then $Stab_{S_n}(a)$ consists of only even permutations, hence τ and σ have the same parity. So in this case a and b are cocnjugate if and only if τ is even.

First we claim that:

Theorem 1. *If the number of distinct integers in a is less than $n-1$, then a and b are conjugate in A_n.*

Proof. Since a and b are conjugate in S_n, then there exists an permutation $\tau \in S_n$ such that $\tau a \tau^{-1} = a$. By hypothesis, we have at least two distinct integers not in a, say p and q. then $\sigma = \tau(pq)$ has different parity with τ and $\sigma \in Stab_{S_n}(a)$. Hence proves our theorem. □

Lemma 2. *If the number of distinct integers in a is greater than or equal to $n-1$, then all permutations in $Stab_{S_n}(a)$ are even if and only if a consists of cycles of distinct odd length.*

Proof. Let a consists of cycles of distinct odd length, say $a = (s_0, s_1, \ldots, s_{s-1})(t_0, t_1, \ldots, t_{t-1}) \cdots (q_0, q_1, \ldots, q_{q-1})$, where $s, t, \ldots, q$ are distinct odd integers. If $\sigma \in Stab_{S_n}(a)$, namely $\sigma a \sigma^{-1} = a$, then $\sigma(s_0, s_1, \ldots, s_{s-1})\sigma^{-1} = (s_0, s_1, \ldots, s_{s-1})$. This means the effect of σ acting on $(s_0, s_1, \ldots, s_{s-1})$ is "pushing forward" each integer x steps in the cycle, namely $s_i \mapsto s_{i+x}$, where $0 \le x \le s-1$, and all subscripts are taken modulo s.

When only considering $(s_0, s_1, \ldots, s_{s-1})$, we may assume without loss of generality that σ consists of integers in $(s_0, s_1, \ldots, s_{s-1})$. Then σ consists of cycles of the same odd length (e.g. when $x \mid s$, $\sigma = (s_0, s_x, \ldots, s_{(k-1)x})(s_1, s_{1+x}, \ldots, s_{1+(k-1)x}) \cdots (s_{x-1}, s_{2x-1}, \ldots, s_{kx-1})$), which means σ is even. The effect of σ acting on other cycles follows in the same fashion. So σ consists of multiple such permutations and no other cycles of length greater than 1 can be added to this permutation since there is at most one integer unused. Hence σ is even.

For the converse, it is equivalent to say that if a doesn't consist of cycles of distinct odd length, then there exists at least one odd permutation in $Stab_{S_n}(a)$. There are two cases.

- If $(p_0, p_1, \ldots, p_{p-1})$ is a cycle of even length in a, then $\sigma = (p_0, p_1, \ldots, p_{p-1})$ is an odd permutation such that $\sigma a \sigma^{-1} = a$.
- If $(e_0, e_1, \ldots, e_{m-1})$ and $(f_0, f_1, \ldots, f_{m-1})$ are two cycles of the same odd length in a, then $\sigma = (e_0, f_0)(e_1, f_1) \cdots (e_{m-1}, f_{m-1})$ is the desired odd permutation such that $\sigma a \sigma^{-1} = a$. □

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Hence we've proved:

Theorem 3. *Let a and b be two conjugate even permutations in S_n and τ any permutation such that $\tau a \tau^{-1} = b$. Then a and b aren't conjugate in A_n if and only if the number of distinct integers in a is greater than or equal to $n-1$, τ is odd, and if a consists of cycles of distinct odd length.*

E-mail address: 2h1p1n9.xu@gmail.com

School of Mathematical Science, Shanxi University, Taiyuan 030006, PR China